Fluctuations as stochastic deformation

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A notion of stochastic deformation is introduced and the corresponding algebraic deformation procedure is developed. This procedure is analogous to the deformation of an algebra of observables like deformation quantization, but for an imaginary deformation parameter (the Planck constant). This method is demonstrated on diverse relativistic and nonrelativistic models with finite and infinite degrees of freedom. It is shown that under stochastic deformation the model of a nonrelativistic particle interacting with the electromagnetic field on a curved background passes into the stochastic model described by the Fokker-Planck equation with the diffusion tensor being the inverse metric tensor. The first stochastic correction to the Newton equations for this system is found. The Klein-Kramers equation is also derived as the stochastic deformation of a certain classical model. Relativistic generalizations of the Fokker-Planck and Klein-Kramers equations are obtained by applying the procedure of stochastic deformation to appropriate relativistic classical models. The analog of the Fokker-Planck equation associated with the stochastic Lorentz-Dirac equation is derived too. The stochastic deformation of the models of a free scalar field and an electromagnetic field is investigated. It turns out that in the latter case the obtained stochastic model describes a fluctuating electromagnetic field in a transparent medium.

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I. INTRODUCTION

Stochastic equations are usually considered as an effective tool that is able to take into account infinitely many factors acting on an open system. Generally, they are obtained by adding to the classical equations of motion of the system a random force (noise) with some probability distribution law. In spite of this common view we shall develop here a slightly different technique inspired by an algebraic approach to quantum mechanics. As is known, quantum fluctuations or quantum mechanics itself can be regarded as resulting from a deformation of underlying classical mechanics or, more precisely, from an algebraic deformation of the Poisson structure on the phase space of the classical system. This approach, known as deformation quantization, was initiated in the seminal papers [1]. Since the quantum fluctuations arise from the deformation of the Poisson structure, it is reasonable to pose the question of whether the stochastic fluctuations could be reproduced in such a way. In this paper we give an affirmative answer on this question and sustain it by diverse examples.

A similarity between equations of quantum and stochastic mechanics was perceived by many authors starting from Schrödinger himself [2]. Much afterwards this similarity was brought in an almost perfect accordance by Zambrini in his works [3] on Euclidean quantum mechanics [4]. However, the key point of these works as well as the works of other authors [5–9] was an analytic continuation to imaginary time and not the deformation of the Poisson structure. Further we shall see that in the relativistic case this slight disagreement is not a matter of interpretation but gives inequivalent results.

The principal point of this paper is that we can obtain stochastic mechanics deforming the associative and commutative product of the algebra of classical observables (the smooth functions over the phase space) along the same lines as in the deformation quantization procedure. However, contrary to an ordinary quantum mechanics with the real deformation parameter (the Planck constant) we deform the product by an imaginary deformation parameter. By analogy we shall refer to this procedure as stochastic deformation [10]. The deformation parameter, which we denote by $\hbar$, characterizes the "openness" of the system and a variance of random forces is proportional to it. It should be mentioned that the Hamiltonian generating the dynamics of stochastic system may take forms unexpected at first glance, because the classical limit $\hbar \to 0$ of stochastic mechanics is not an ordinary classical mechanics. In this limit we obtain some classical mechanics where the momenta should be regarded as systematic forces (or proportional to them) acting on the system, for this limit is equivalent to a treatment of the system in the equilibrium state when stochastic fluctuations vanish and the effect of systematic forces is balanced by a dissipation. The expansion of the pq symbol of the Hamiltonian in momenta is closely related to the Kramers-Moyal expansion and the coefficients of this expansion are connected with the cumulants of the probability density function of a noise. In the simplest case of a linear symplectic space the quadratic in momenta Hamiltonian corresponds to a Gaussian noise.

To avoid misunderstanding we note in advance that the result of stochastic deformation of a classical model, that we call stochastic mechanics, is not Nelson's stochastic mechanics [11] developed to give a stochastic interpretation of quantum mechanics. As a practical matter, our approach can be regarded as a gauged version of the operator approach to stochastic mechanics expounded in [6–9,12,13]. Such a formulation offers the advantage that one can apply the developed methods of quantum mechanics and field theory to sto-

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chastic mechanics almost without any changing. In this context it is sufficient to mention, for instance, the models with gauge symmetries. The methods of quantization of systems with constraints (see for an introduction [14–16]) adapted to stochastic deformation give the technique of generation of stochastic models respecting not merely the global symmetries of the initial classical system but the gauge symmetries as well. In other words, these methods allow us to introduce the noise to physical degrees of freedom that are not expressed in an explicit form. We shall consider several models with gauge symmetries in this paper.

The paper is organized as follows. We start our investigation with a formulation of general rules of stochastic deformation by the example of a linear symplectic space (Sec. II). In developing this scheme we follow the central proposal of deformation quantization and try to devise the procedure in a more algebraic way. We do not deal with the linear space of states and its dual but with the algebra of operators on them which are recognized as stochastic observables. A state of a stochastic system is specified by an analog of the density operator. For pure states it is useful to realize the algebra of observables in some linear space, at that we construct this space by the action of creation operators on the vacuum state. This standpoint allows us to calculate averages and make some proofs by means of the basic relation of the Heisenberg-Weyl algebra only. Generally speaking, the realization of some operators arising on intermediate steps of calculations can lead to divergent integrals or series that cancel each other in the final result. Then the algebraic approach can be considered as some regularization or prescription to handle these singular integrals. Besides, in this algebraic way we can establish stochastic mechanics for systems with nonlinear phase space such as symplectic or even nonregular Poisson manifolds. Explicit constructions of the deformed product (star product) of the algebra of smooth functions on such manifolds are given in [17,18].

In conclusion of Sec. II we relate the mechanics obtained from stochastic deformation and ordinarily formulated stochastic mechanics using a path-integral representation of the transition probability and the Langevin equations associated with it. There we present the formal relation only. The existence problems are left beyond the scope of the paper. The very method of stochastic deformation implies that the partial differential equations of the Fokker-Planck type are primary for this framework, while the stochastic equations of the Langevin type are only used to give a more lucid physical interpretation to the obtained equations. At the same time the form of the Hamiltonian generating an evolution is related to the probability density function of the noise. Thereby the information about the probability distribution law for the noise enters to the stochastic deformation procedure.

In Sec. III we consider several significant models revealing the key features of the developed formalism. All of them are divided into three categories. The first class exposed in Sec. III A is constituted by the models of a nonrelativistic particle under the influence of a random force. It is a classical subject of stochastic mechanics and we consider these models to include them into a general scheme and gain an experience required in understanding of subsequent sections. Namely, in this section we study the stochastic deformation of the models of a nonrelativistic particle coupled to the electromagnetic and gravitational fields as well as the stochastic deformation of the model leading to the Klein-Kramers equation. Under stochastic deformation the first two models result in the stochastic systems described by the Fokker-Planck equations with trivial and nontrivial diffusion tensors, respectively, the diffusion tensor being the inverse metric [19].

In Sec. III B we extend our analysis to relativistic models with constraints. We investigate three stochastic models of a relativistic particle affected by a random force. The first model is the stochastically deformed model of a relativistic particle interacting with the electromagnetic field. We show that such a model is described by a relativistic equation which generalizes the Fokker-Planck equation in the same sense as the Klein-Gordon equation generalizes the Schrödinger equation. For a constant systematic force acting on the particle the obtained equation is in one-to-one correspondence with the relativistic diffusion equation [20]. Making use of the descent method [21,22] we derive a path-integral representation of the transition probability, where it turns out that such a relativistic Fokker-Planck equation corresponds to a relativistic particle under the influence of a non-Gaussian noise. The second model examined in this section is a relativistic generalization of the model leading under stochastic deformation to the Klein-Kramers equation. The respective relativistic equation proves to describe a relativistic diffusion studied in [23–28]. Here we also obtain a path-integral representation of the transition probability both in the gauge of a laboratory time and in the proper time gauge. The third model of this section is related to stochastic mechanics of a relativistic charged particle with the radiation reaction taken into account, i.e., we derive in this section a relativistic analog of the Fokker-Planck equation associated with the stochastic Lorentz-Dirac equation. It looks very likely that the procedure developed in these sections is straightforwardly generalized to stochastic reparametrization invariant relativistic ordinary differential equations of an arbitrary order.

Section III C is devoted to stochastic deformation of relativistic field theories. We only touch the problem and consider free models of the scalar and electromagnetic fields. Under the assumptions of causality and relativistic invariance we derive the propagators for these models. In the case of the electromagnetic field the obtained propagator coincides with the well-known correlator of the electromagnetic fields in a transparent medium [29]. Besides the last model illustrates how the Becchi-Rouet-Stora-Tyutin (BRST) quantization technique can be applied to stochastic mechanics.

Thus we see that the stochastic deformation method covers a considerable part of stochastic physics and apparently any other stochastic model can be formulated in terms of this unifying algebraic approach. The first model in Sec. III B shows that the non-Markovian processes can be obtained by stochastic deformation as well. The non-Markovian processes have to be described by the models with constraints. Inasmuch as we regard in this paper many topics of theoretical physics the reference list is, of course, incomplete. I tried to make references to the basic works known to me and to the works that might be useful to the reader in understanding and further developing the scheme evolved here.

We use the following notation and conventions. The system of units is chosen so that the velocity of light $c=1$. Greek letters denote space-time indices and Latin indices indicate the spatial components of tensors. Sometimes we shall use boldface characters to denote the spatial part of coordinates. Einstein's summation rule is assumed unless otherwise stated. Usually overdots will denote a differentiation with respect to time. In the sections regarding nonrelativistic models d is a dimension of the configuration space with the coordinates x and indices are risen and lowered by the metric tensor $\eta_{\mu\nu}=\mathrm{diag}(-1,1,\ldots,1)$, while for the relativistic models d is a dimension of the space-time, x is the space-time coordinates, and the metric tensor $\eta_{\mu\nu}$ reads as $\mathrm{diag}(1,-1,\ldots,-1)$.

II. THE RULES OF STOCHASTIC DEFORMATION

In this section we formulate the rules of stochastic deformation and establish the relation between a stochastically deformed model and a certain stochastic mechanics [30]. Consider a classical system on the linear symplectic manifold M with canonical coordinates x^i and p_j:

$$\{x^i,p_j\}=\delta^i_j, \quad i,j=\overline{1,d}, \tag{1}$$

where d is a dimension of the configuration space and curly brackets denote the Poisson brackets. The algebra of classical observables is the real commutative associative algebra of smooth functions on M. The evolution is generated by the observable $H(t,q,p)$ that is the Hamilton function.

Then we deform the algebra of classical observables in the manner of deformation quantization but with an imaginary deformation parameter (the Planck constant) such that

$$[\hat{x}^i,\hat{p}_j]=\hbar\delta^i_j, \quad \hbar>0. \tag{2}$$

Hereinafter we denote elements of the deformed algebra by hats and imply the Weyl-Moyal star-product [31]

$$\hat{f}\hat{g}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\hbar}{2}\right)^n\omega^{a_1b_1}\cdots\omega^{a_nb_n}\partial_{a_1\cdots a_n}f(z)\partial_{b_1\cdots b_n}g(z), \tag{3}$$

where $z\equiv(x,p)$, $a_n,b_n=\overline{1,2d}$, the functions $f(z)$ and $g(z)$ are the Weyl symbols of the corresponding elements of the deformed algebra, ω^{ab} is the inverse to the symplectic two-form ω_{ab}. Recall that the Weyl-ordered operator for a monomial in momenta symbol looks like

$$V^{i_1\cdots i_n}(x)p_{i_1}\cdots p_{i_n}\to\frac{1}{2^n}\sum_{k=0}^{n}C^k_n\hat{p}_{i_1}\cdots\hat{p}_{i_k}V^{i_1\cdots i_n}(\hat{x})\hat{p}_{i_{k+1}}\cdots\hat{p}_{i_n}, \tag{4}$$

where C^k_n are the binomial coefficients. The physical meaning of the constant $\hbar$ in formula (2) will be elucidated below. Roughly, $\hbar$ characterizes a variance of stochastic forces acting on a classical system.

Let us give a linear functional Tr on the deformed algebra mapping to real numbers and vanishing on commutators, viz.

$$\mathrm{Tr}(\hat{f}\hat{g})=\mathrm{Tr}(\hat{g}\hat{f}), \quad \forall\hat{f},\hat{g}, \tag{5}$$

which we shall call the trace. An explicit formula for the trace of the element $\hat{f}$ has the form

$$\mathrm{Tr}\ \hat{f}=\int\frac{d^dxd^dp}{(2\pi\hbar)^d}f(x,ip)\Rightarrow\mathrm{Tr}(\hat{f}\hat{g})=\int\frac{d^dxd^dp}{(2\pi\hbar)^d}f(x,ip)g(x,ip). \tag{6}$$

Then we define a complete set of elements $\{\hat{\rho}_\lambda\}$ of the deformed algebra by the properties

$$1.\quad\sum_\lambda\hat{\rho}_\lambda=\hat{1},$$

$$2.\quad\mathrm{Tr}\ \hat{\rho}_\lambda=1,$$

$$3.\quad\hat{\rho}_\lambda\hat{\rho}_{\lambda'}=\delta_{\lambda\lambda'}\hat{\rho}_\lambda. \tag{7}$$

Consider a class of the complete sets related to each other by similarity transformations, i.e., two complete sets $\{\hat{\rho}_\lambda\}$ and $\{\hat{\sigma}_\mu\}$ are in the same class if there exists an invertible element $\hat{U}$ in the algebra spanned on the generators $\hat{x}^i$ and $\hat{p}_j$ such that

$$\hat{\sigma}_{f(\lambda)}=\hat{U}^{-1}\hat{\rho}_\lambda\hat{U}, \tag{8}$$

where f is a bijection and $\hat{U}$ does not depend on λ. We choose the class which contains the complete set $\{\hat{\rho}_x\}$ corresponding to the x representation

$$\hat{\rho}_x=\delta^d(\hat{x}^i-x^i)=\int\frac{d^dp}{(2\pi\hbar)^d}\exp\left[\frac{i}{\hbar}p_i(\hat{x}^i-x^i)\right]. \tag{9}$$

This class also contains the complete sets associated with any Lagrangian section of the symplectic space M obtained from the coordinate Lagrangian section by a linear symplectic transformation. The element $\hat{U}$ realizing a similarity transformation is a solution of the equation

$$\hbar\dot{\hat{U}}(t)=\frac{1}{2}\hat{z}^a\omega_{ab}A^b_c\hat{z}^c\hat{U}(t), \quad \hat{U}(0)=\hat{1}, \tag{10}$$

for appropriate t, where the matrix A^a_b belongs to the Lie algebra of the symplectic group.

We say that the element $\hat{T}$ of the deformed algebra is a stochastic observable if there exists a complete set $\{\hat{t}_\lambda\}$ from the chosen class such that

$$\hat{T}=\sum_\lambda T(\lambda)\hat{t}_\lambda, \tag{11}$$

where $T(\lambda)$ is a certain real-valued function. In other words the stochastic observables should be diagonalizable in the chosen class.

The state of the stochastic system is characterized by the observable $\hat{\rho}$ with a unit trace:

$$\text{Tr } \hat{\rho} = 1. \tag{12}$$

The pure state is specified by an additional idempotency requirement

$$\hat{\rho}^2 = \hat{\rho}. \tag{13}$$

The average of an observable $\hat{T}$ over a state $\hat{\rho}$ is defined as

$$\langle \hat{T} \rangle := \text{Tr}(\hat{\rho}\hat{T}). \tag{14}$$

In particular, the average over the state $\hat{\rho}$ of a certain pure state $\hat{\rho}_\lambda$ gives the probability to find a stochastic system in this pure state.

The dynamics of a stochastic system in the state $\hat{\rho}$ are generated by the Weyl-ordered operator $\hat{H}$ corresponding to the Hamilton function $H(t,x,p)$ and described by the von Neumann equation

$$\hbar \dot{\hat{\rho}} = [\hat{H}, \hat{\rho}], \tag{15}$$

whence the "Euclidean" Heisenberg equation for averages follows

$$\hbar \frac{d}{dt} \langle \hat{T} \rangle = \langle \hbar \partial_t \hat{T} + [\hat{T}, \hat{H}] \rangle. \tag{16}$$

The evolution defined in this way maps observables into observables. Besides, it follows from Eq. (15) that the evolution of a probability density function to find a system in the pure state $\hat{\rho}_\lambda$ obeys the equation

$$\hbar \frac{d}{dt} \langle \hat{\rho}_\lambda \rangle_{\hat{\rho}} = \text{Tr}(\hat{\rho}_\lambda [\hat{H}, \hat{\rho}]), \tag{17}$$

which is nothing but the Fokker-Planck equation. The conservation of the total probability is a consequence of the trace property.

Now we are in position to touch the problem of an arbitrariness in a definition of the star-product related to the ordering prescription. This arbitrariness at least for a linear symplectic space does not affect averages and their evolution, but as in quantum mechanics when we convert one star-product to another some observables become nondiagonalizable in a class of complete sets corresponding to the new star product. Anyway, different orderings for the same physical observable or the Hamiltonian just result in different definitions of the quantities measured in stochastic mechanics (correlators) in terms of the coefficients of an expansion of the observable and the Hamiltonian in momenta.

Since the phase space of the given classical system is a linear symplectic space it is useful to realize the deformed algebra by operators acting in some linear subspace V of the linear space of smooth functions on the configuration space. We introduce the Dirac notations for elements from the linear space V and its dual:

$$|\psi\rangle \in V, \quad \langle \varphi| \in V^*, \tag{18}$$

where V^* is the linear space of linear functionals on V. Operators acting in the space V are naturally translated to operators acting in the dual space V^*. The eigenvectors from the space V and V^* of the operator $\hat{T}$ corresponding to the eigenvalue t we denote by

$$|T=t\rangle, \text{ or } |t\rangle, \quad \text{and} \quad \langle T=t|, \text{ or } \langle t|,$$

$$\hat{T}|t\rangle = t|t\rangle, \quad \langle t|\hat{T} = \langle t|t. \tag{19}$$

For the given left and right vectors (vacua) $|x=0\rangle$ and $\langle x=0|$ we construct the eigenvectors of the operator $\hat{x}$

$$|x=a\rangle = e^{1/\hbar a^i \hat{p}_i}|x=0\rangle, \quad \langle x=a| = \langle x=0|e^{-1/\hbar a^i \hat{p}_i}. \tag{20}$$

Their inner product possesses the properties

$$f(x)\langle x|x'\rangle = f(x')\langle x|x'\rangle, \quad \forall f \in V,$$

$$\langle x+\lambda|x'+\lambda\rangle = \langle x|x'\rangle, \quad \forall \lambda, \tag{21}$$

and, consequently, is equal up to a constant factor to the δ function. On setting this constant to unity the element of the complete set (9) can be written as

$$\hat{\rho}_x = |x\rangle\langle x|. \tag{22}$$

In the same manner we can prove that an element of a complete set associated with any other Lagrangian section has the form (22) with appropriate eigenvectors. The linear space V is spanned on the vectors obtained from $|x\rangle$ by an action of the various operators $\hat{U}$ realizing the similarity transformation in the chosen class of complete sets.

Thus for the pure state

$$\hat{\rho} = |\psi\rangle\langle\varphi|, \quad \langle\varphi|\psi\rangle = 1, \tag{23}$$

the probability to find a system in the point x at the time t takes the form

$$\rho(t,x) = \langle x|\psi\rangle\langle\varphi|x\rangle. \tag{24}$$

Now it is easy to obtain a formal solution to the Fokker-Planck equation (17). Its fundamental solution or, in physical terms, the transition probability, can be represented in two ways [32]:

$$G(t',x';t,x) := \frac{\langle x'|\hat{U}_{t',t}|x\rangle\langle\varphi|\hat{U}^{-1}_{t',t}|x'\rangle}{\langle\varphi|x\rangle} = \frac{\langle x'|\hat{U}_{t',t}|\psi\rangle\langle x|\hat{U}^{-1}_{t',t}|x'\rangle}{\langle x|\psi\rangle}, \tag{25}$$

where $\hat{U}_{t',t}$ is the evolution operator obeying the equations

$$\hbar \partial_{t'} \hat{U}_{t',t} = \hat{H}(t')\hat{U}_{t',t}, \quad \hat{U}_{t,t} = \hat{1}. \tag{26}$$

A convolution of Eq. (25) with Eq. (24) gives the probability density function at the moment t'. To provide linearity of Eqs. (17) with respect to $\rho(t,x)$ we have to claim that one of the kernels (25) is independent of $\rho(t,x)$. Therefore either $\langle\varphi|x\rangle$ or $\langle x|\psi\rangle$ is independent of $\rho(t,x)$.

Consequently, the density operator (23) in the x representation looks like

$$\langle x|\psi\rangle\langle\varphi|y\rangle = e^{1/\hbar[S(t,y)-S(t,x)]}\rho(t,x),$$

$$\langle x|\psi\rangle\langle\varphi|y\rangle = e^{1/\hbar[\tilde{S}(t,y)-\tilde{S}(t,x)]}\rho(t,y), \tag{27}$$

for the first and second cases, respectively. Here we introduce $S(t,x)$ which is the analog of a quantum mechanical phase [3,33]. This function can have discontinuities or even be complex but its gradient contributing to the observable averages should be real and have removable discontinuities only. The first representation in Eq. (25) for the fundamental solution is a forward evolution operator twisted by $e^{1/\hbar S(t,x)}$ and it takes the form of a forward transition probability, while the second representation of the fundamental solution has the form of a backward transition probability for the Fokker-Planck equation.

These transition probabilities depend on the phase having different definitions for the first and second cases (27). For the given initial probability density function one can choose the initial phase in such a way that the forward transition probability will give the same probabilities as the process generated by the backward transition probability with the same initial probability density function. The phase is a new "degree of freedom" as against classical mechanics and the above mentioned flexibility of the formalism is only concerned with its interpretation in terms of stochastic mechanical quantities. For the forward transition probability with the definition of phase given by the first formula in Eq. (27) this interpretation takes the simplest form and will be discussed below. Henceforth we shall refer to the forward transition probability as the transition probability.

A. Stochastic deformation and Langevin equation

By standard means (see, e.g., [31,34]) we can construct a path-integral representation of the transition probability (25). Notice in advance that we shall not consider here the existence problems of fundamental solutions and their well-defined path-integral representations. The interested reader can consult the references above.

Using the relation between the kernel of the Weyl-ordered operator $\hat{T}$ in the coordinate representation and its symbol

$$\langle x'|\hat{T}|x\rangle = \int \frac{d^dp}{(2\pi\hbar)^d} T\left(\frac{x'+x}{2}, ip\right) e^{-i/\hbar p_i(x'^i-x^i)}, \tag{28}$$

we formally have

$$\begin{aligned}&\langle\varphi(t+dt)|x'\rangle\langle x'|\hat{U}_{t+dt,t}|x\rangle\frac{1}{\langle\varphi(t)|x\rangle}\\&=\langle x'|\exp\left(\frac{dt}{\hbar}\{\hat{H}[t,\hat{x},\hat{p}+\nabla S(t,\hat{x})]+\partial_t S(t,\hat{x})\}\right)|x\rangle\\&=\int\frac{d^dp(t)}{(2\pi\hbar)^d}\exp\left(-\frac{i}{\hbar}\{p_i(t)\dot{x}^i(t)\right.\\&\left.\quad+i[\bar{H}_W(t,\tilde{x}(t),ip(t))+\partial_t S(t,\tilde{x}(t))]\}dt\right),\end{aligned} \tag{29}$$

where $\bar{H}_W(t,x,p)$ is a Weyl-symbol of the Hamiltonian $\hat{H}$ in which the momenta operators $\hat{p}_j$ are replaced by $\hat{p}_j+\partial_j\hat{S}$ and

$$x(t)=x,\quad x(t+dt)=x',\quad \dot{x}(t):=[x(t+dt)-x(t)]/dt,$$

$$\tilde{x}(t):=[x(t+dt)+x(t)]/2.$$

As long as the transition probability (25) possesses the defining property of the Markov process we can cut the time interval $[t,t']$ into pieces for which formula (29) makes sense and then integrate over intermediate positions. As a result the fundamental solution to the Fokker-Planck equation (17) takes the form

$$\begin{aligned}G(t',x';t,x)=&\int\prod_{\tau\in(t,t')}d^dx(\tau)\prod_{\tau\in[t,t')}\frac{d^dp(\tau)}{(2\pi\hbar)^d}\\&\times\exp\left(-\frac{i}{\hbar}\int_t^{t'-d\tau}d\tau\{p_i(\tau)\dot{x}^i(\tau)\right.\\&\left.+i[\bar{H}_W(\tau,\tilde{x}(\tau),ip(\tau))+\partial_\tau S(\tau,\tilde{x}(\tau))]\}\right).\end{aligned} \tag{30}$$

Notice that the main contribution to the transition probability (30) is made by paths approximating a classical trajectory. This becomes manifest if one makes the change of variables $p_j\to p_j-\partial_j S$ and neglects stochastic corrections.

On integrating Eq. (30) over momenta we arrive at the functional integral with the expression in the exponent taking the form of the Stratonovich-type stochastic integral (see, e.g., [8]). It is a consequence of the use of Weyl symbol for the evolution operator. If we had chosen the pq symbol, we would have obtained the Ito-type stochastic integral. Namely, in that case the transition probability (30) becomes

$$\begin{aligned}G(t',x';t,x)=&\int\prod_{\tau\in(t,t')}d^dx(\tau)\prod_{\tau\in[t,t')}\frac{d^dp(\tau)}{(2\pi\hbar)^d}\\&\times\exp\left(-\frac{i}{\hbar}\int_t^{t'-d\tau}d\tau\{p_i(\tau)\dot{x}^i(\tau)\right.\\&\left.+i[\bar{H}(\tau,x(\tau),ip(\tau))+\partial_\tau S(\tau,x(\tau))]\}\right),\end{aligned} \tag{31}$$

where $\bar{H}(t,x,p)$ is a pq symbol of the Hamiltonian operator with the momenta $\hat{p}_j+\partial_j\hat{S}$. Certainly, the transition probabilities (30) and (31) are equal to each other as they are different symbols of the same evolution operator.

Let us now provide an interpretation of the above stochastic mechanics in terms of the Langevin equation

$$\dot{x}^i(\tau)-\nu^i(\tau)=0. \tag{32}$$

To this end we insert the δ function with the left-hand side (LHS) of Eq. (32) in its argument into the path integral (31) and integrate it over $\nu^i(\tau)$. Then

$$F(\nu,\tau,x) := \int \frac{d^d p}{(2\pi\hbar)^d} e^{-i/\hbar d\tau p_i \nu^i} \times \exp\left(\frac{d\tau}{\hbar}[\bar{H}(\tau,x,ip) + \partial_\tau S(\tau,x)]\right) \qquad (33)$$

can be interpreted as a probability density function of the noise $\nu^i(\tau)$ at the time slice τ, the Langevin Eq. (32) being understood in the Ito sense. The function

$$\Phi(\lambda,\tau,x) := \int d^d \nu e^{i\lambda_i \nu^i} F(\nu,\tau,x) = (d\tau)^{-d} \exp\left\{\frac{d\tau}{\hbar}\left[\bar{H}\left(\tau,x,i\lambda\frac{\hbar}{d\tau}\right) + \partial_\tau S(\tau,x)\right]\right\} \qquad (34)$$

is known as the characteristic function and the coefficients of the Taylor series in $i\lambda$ of the expression in the exponent are called the cumulants. For example, the second cumulant is proportional to the inverse metric (inverse mass matrix) in the Hamiltonian $\bar{H}(t,x,p)$ multiplied by the deformation parameter $\hbar$. The second cumulant is equal to the mean-squared deviation which is why we claimed that $\hbar$ characterizes the variance of stochastic forces acting on the system.

Now it is easy to see that the Hamiltonian most quadratic in momenta leads to the Langevin equation with a noise whose probability density function is a product of delta and Gaussian functions. The Hamiltonians depending on higher powers of momenta correspond to a non-Gaussian noise. If the Hamiltonian is analytic in momenta then in the classical limit, $\hbar \to 0$, only the first cumulant survives and the transition probability (31) becomes Liouvillian. The zeroth cumulant is a mere normalization factor and, as follows from the normalization condition on Eq. (31), is equal to $-d \ln d\tau$.

The above interpretation in terms of the Langevin equation works well when the probability density (33) of noise is a positive and normalizable function. If this function takes negative values then the transition probability (31) also possesses negative values for some values of its arguments. However, it does not automatically imply that the considered stochastic mechanics is unphysical. For example, this fact may signify that the δ-localized probability density functions are not well defined for such a system from the physical point of view. At the same time there may exist a class of probability density functions for which a convolution with the transition probability gives reasonable results. We shall encounter the problem of negative probabilities in considering a relativistic generalization of the Fokker-Planck equation in Sec. III B 1.

III. EXAMPLES

A. Nonrelativistic particle

In this subsection we consider the stochastic deformation of three nonrelativistic models: a particle interacting with the electromagnetic field, the same model on a curved background, and the model leading to the Klein-Kramers equation. For these models we obtain a path-integral representation of the transition probabilities and the associated Langevin equations. Some simple applications of the developed scheme are also demonstrated.

1. Fokker-Planck equation

According to general rules expounded in the previous section the Weyl-ordered Hamiltonian for a nonrelativistic particle looks like

$$\hat{H} = \frac{(\hat{p}_i - \hat{A}_i)^2}{2m} + \hat{A}^0. \qquad (35)$$

It is convenient to realize the operators $\hat{x}^i$ and $\hat{p}_j$ in the linear space V of functions on the configuration space in the following way:

$$\hat{x}^i = x^i, \quad \hat{p}_i = -\hbar\partial_i. \qquad (36)$$

Then for the pure state of the form (23) the von Neumann equation (15) reduces to the two "Euclidean" Schrödinger equations

$$\hbar\partial_t\psi(t,x) = \left[\frac{(\hat{p}_i - A_i)^2}{2m} - A_0\right]\psi(t,x),$$

$$\hbar\partial_t\varphi(t,x) = -\left[\frac{(\hat{p}_i + A_i)^2}{2m} - A_0\right]\varphi(t,x), \qquad (37)$$

provided that the standard inner product is understood. For a correct stochastic interpretation the functions $\psi(t,x)$ and $\varphi(t,x)$ have to be both positive (or both negative). The fields A_μ are the gauge fields, which we shall call the electromagnetic fields. Their physical meaning is obvious from the general considerations of the previous section. Namely, introducing the phase, $S := \hbar \ln \varphi$, we see from Eq. (34) that the quantity

$$\partial_i S - A_i \qquad (38)$$

is proportional to the first cumulant of the probability density function of noise (33) whereas the first cumulant is equal to the expectation value of a random variable. We dwell on the interpretation of A_μ a bit later.

The Schrödinger equations (37) are invariant under the following gauge transformations:

$$\psi(t,x) \to \psi(t,x)e^{-\xi(t,x)}, \quad \varphi(t,x) \to \varphi(t,x)e^{\xi(t,x)},$$

$$A_\mu(t,x) \to A_\mu(t,x) + \partial_\mu\xi(t,x). \qquad (39)$$

In particular, these transformations do not change the probability density function. The conserved four-current corresponding to the gauge transformations (39) is given by

$$j^\mu = \left(\varphi\psi, \frac{1}{2m}[\varphi(\hat{p}^i - A^i)\psi - \psi(\hat{p}^i + A^i)\varphi]\right). \qquad (40)$$

The system of equations (37) is Lagrangian with the Hamiltonian action of the form

$$S_H[\varphi,\psi] = \int dt\langle\varphi|\hbar\partial_t - \hat{H}|\psi\rangle, \tag{41}$$

that is, the fields $\psi(t,x)$ and $\varphi(t,x)$ are canonically conjugate with respect to the Poisson bracket.

On substituting the definition of phase (27), the system of evolutionary equations (37) can be rewritten in the equivalent form

$$\partial_t\rho = -\partial^i\left[-\frac{\hbar}{2m}\partial_i\rho + \frac{\partial_i S - A_i}{m}\rho\right],$$

$$\partial_t S - A_0 + \frac{(\partial_i S - A_i)^2}{2m} = -\frac{\hbar}{2m}\partial^i(\partial_i S - A_i). \tag{42}$$

The first equation is the Fokker-Planck equation, which is the divergence of the four-current (40), while the second equation can be referred to as the quantum Hamilton-Jacobi equation [33,35]. Now it is evident that if one neglects stochastic corrections then the initially δ-shaped probability density function $\rho(t,x)$ keeps its own form and propagates as a classical charged particle in the electromagnetic field [36] with particle's momentum $\partial_i S(t,x) - A_i(t,x)$.

Taking into account the stochastic corrections we see that the more the probability density function is localized, the higher is the probability flow resisting localization. It can be perceived, for example, from the stochastic analog of the quantum mechanical uncertainty relation

$$\langle(x^i)^2\rangle\langle(p^i_{os})^2\rangle \geq \frac{\hbar^2}{4}, \tag{43}$$

where summation is not understood, $p^i_{os} := -\hbar\partial^i \ln \rho^{1/2}$ is the osmotic momentum. This relation is easily deduced from the inequality

$$\int d^d x[(\xi x^i - \hbar\partial_i)\rho^{1/2}]^2 \geq 0, \quad \forall\, \xi \in \mathbb{R}, \tag{44}$$

under the assumption that $\rho(x)$ tends to zero at spatial infinity faster than x^{-2}.

To find the first stochastic correction to the classical equations of motion we use the Heisenberg equations (16) for averages of the operators $\hat{x}^i$ and $\hat{p}_j$,

$$m\frac{d}{dt}\langle x_i\rangle = \langle\hat{p}_i - A_i\rangle = \langle\partial_i S - A_i\rangle,$$

$$\frac{d}{dt}\langle\hat{p}_i\rangle = -\langle\partial_i A^0\rangle + \frac{1}{2m}\langle\partial_i A_j(\hat{p}_j - A_j) + (\hat{p}_j - A_j)\partial_i A_j\rangle, \tag{45}$$

whence

$$m\frac{d^2}{dt^2}\langle x_i\rangle = \langle E_i\rangle + \frac{1}{m}\varepsilon_{ijk}\langle(\partial_j S - A_j)H_k\rangle + \frac{\hbar}{2m}\varepsilon_{ijk}\langle\partial_j H_k\rangle. \tag{46}$$

In the case where $\rho(t,x)$ is sufficiently localized compared to the characteristic scale of variations of the electromagnetic fields the angle brackets can be carried through the electromagnetic fields to obtain a closed system of evolutionary equations on the average position. They are the Newton equations with the stochastic correction.

Now we return to the interpretation of the gauge fields A_μ. Recall that the expectation value of noise in the Langevin equation (32) is called a systematic drift. In our case the systematic drift is equal to

$$f_i(t,x) := \partial_i S(t,x) - A_i(t,x), \tag{47}$$

where we set $m=1$. Therefore to satisfy the second equation in Eq. (42) we have to take

$$A_0 - \partial_t S = \frac{1}{2}(f^2 + \hbar\partial_i f^i). \tag{48}$$

The system of equations (47) and (48) with respect to $A_\mu(t,x)$ and $S(t,x)$ obviously admits a solution for any systematic drift $f_i(t,x)$. The fields $A_\mu(t,x)$ and $S(t,x)$ are not uniquely defined by these equations and the arbitrariness in their definition is equivalent to the arbitrariness of a gauge. In particular, in the "unitary" gauge $S(t,x)=0$ the gauge fields A_i are the components of the systematic drift with an opposite sign.

The Newton equations (46) for the average position of the particle in the representation (47) and (48) become

$$\frac{d}{dt}\langle x^i\rangle = \langle f^i\rangle, \quad \frac{d^2}{dt^2}\langle x^i\rangle = \langle(\partial_t + f^j\partial_j)f^i\rangle + \frac{\hbar}{2}\langle\Delta f^i\rangle. \tag{49}$$

For example, if $f^i(t,x)$ is the velocity field of an incompressible viscous fluid with the specific pressure p and kinematic viscosity ν (see, e.g., [37]) then the second equation in Eq. (49) is replaced by

$$\frac{d^2}{dt^2}\langle x_i\rangle = -\langle\partial_i p\rangle + (\nu + \hbar/2)\langle\Delta f_i\rangle, \tag{50}$$

i.e., the acceleration of mean position of the Brownian particle is the same as for the particle which is not influenced by stochastic forces but entrained by a fluid with a higher viscosity.

To gain a better physical insight into the stochastically deformed model of a nonrelativistic particle we construct the functional integral representation (30) of the transition probability. The Weyl symbol of the Hamiltonian with the momenta $\hat{p}_j + \partial_j S$ arising in formula (31) is

$$\bar{H}(t,x,ip) = \frac{1}{2m}[-p^2 + 2ip^i(\partial_i S - A_i) + (\partial_i S - A_i)^2] + A^0. \tag{51}$$

Substituting this expression into Eq. (30) and integrating over momenta we arrive at

$$G(t',x';t,x) = \int\left(\frac{m}{2\pi\hbar d\tau}\right)^{d/2}\prod_{\tau\in(t,t')}\left(\frac{m}{2\pi\hbar d\tau}\right)^{d/2} d^d x(\tau) \times\exp\left[-\frac{1}{\hbar}\int_t^{t'-d\tau} d\tau\left(\frac{m}{2}\dot{x}^2 + (A_i - \partial_i S)\dot{x}^i - (A^0 + \partial_\tau S)\right)\right], \tag{52}$$

where the functions $A_\mu(t,x)$ and $S(t,x)$ obey the quantum Hamilton-Jacobi equation (42) and are taken at the point $(t,x)=(\tau,\tilde{x}(\tau))$. Since the expression in the exponent is the classical action modulo boundary terms the main contribution to the transition probability is made by the paths approximating a classical trajectory that is in agreement with our general considerations. In the representation (47) and (48) the transition probability (52) reduces to the well-known result

$$G(t',x';t,x) = \int \frac{1}{(2\pi\hbar d\tau)^{d/2}} \prod_{\tau\in(t,t')} \frac{d^d x(\tau)}{(2\pi\hbar d\tau)^{d/2}} \times\exp\left(-\frac{1}{2\hbar}\int_t^{t'-d\tau} d\tau\{[\dot{x}(\tau) - f(\tau,\tilde{x}(\tau))]^2 + \hbar\partial_i f^i(\tau,\tilde{x}(\tau))\}\right). \tag{53}$$

So we have obtained the transition probability by the use of the Weyl symbol of the evolution operator. The transition probability in terms of the pq symbol (31) of the evolution operator is constructed along the same lines and looks like Eq. (53) without the stochastic correction to the action and the midpoint prescription.

In conclusion of this section we briefly discuss how different generalizations of the Fokker-Planck equation of the form (42) can be constructed in the developed framework.

2. Fokker-Planck equation with nontrivial diffusion tensor

At first we consider stochastic deformation of the model (35) on a curved space background with the inverse metric g^{ij}. From the general considerations we know that the inverse metric is proportional to the covariance (the second cumulant) of noise in the Langevin equation (32), while the gauge fields A_i are related to expectation values of the noise. Besides, to provide a convergence of the integral (33) over momenta we have to require that the eigenvalues of the matrix g^{ij} are nonnegative.

In constructing the Hamiltonian by its symbol we follow the simplest prescription [38] based on the use of the exponential map from the tangent bundle to the configuration space generated by the Levi-Civita connection (for more sophisticated methods, see, e.g., [39,40]). In other words, we recognize the momenta $\hat{p}_i$ as the derivatives at the origin of the Riemann normal coordinates and the vectors of the linear space V as the scalar functions on the manifold. Neglecting for a while the gauge fields A_μ we have for the Hamiltonian (35) in the normal coordinates

$$\hat{H}_0 = \frac{1}{8m}(\hat{p}_i\hat{p}_j g^{ij} + 2\hat{p}_i g^{ij}\hat{p}_j + g^{ij}\hat{p}_i\hat{p}_j). \tag{54}$$

Making use of relations between the derivatives of the metric taken at the origin of the normal coordinates and the Riemannian tensor [41,42] we arrive at [43]

$$\hat{H}_0 = \frac{\hbar^2}{2m}\left(\nabla^2 - \frac{R}{12}\right), \tag{55}$$

where R is the scalar curvature and ∇_i is the Levi-Civita connection. Notice that the covariant pq ordering leads to the coefficient 1/3 at the curvature as in [42], the covariant qp ordering gives rise to the vanishing coefficient (minimal coupling) and

$$\hat{H}_0 = \frac{1}{4m}(\hat{p}_i\hat{p}_j g^{ij} + g^{ij}\hat{p}_i\hat{p}_j)$$

corresponds to Eq. (55) with the coefficient 1/6 at the curvature (conformal coupling) [44]. All of these prescriptions result effectively in a variation of the coefficient at the scalar curvature.

Thus the Hamiltonian for a particle interacting with the gauge fields A_μ on a curved space background looks like

$$\hat{H} = \frac{1}{2m}(\hbar\nabla_i + A_i)^2 - \frac{\hbar^2 R}{24m} + A^0. \tag{56}$$

Substituting the phase definition Eq. (27) into the Schrödinger equations following from the action (41) we obtain [19]

$$g^{-1/2}\partial_t(g^{1/2}\rho) = -\nabla^i\left[-\frac{\hbar}{2m}\partial_i\rho + \frac{\partial_i S - A_i}{m}\rho\right],$$

$$\partial_t S - A_0 + \frac{(\partial_i S - A_i)^2}{2m} = -\hbar\partial_t \ln g^{1/2} - \frac{\hbar}{2m}\nabla^i(\partial_i S - A_i) + \frac{\hbar^2 R}{24m}, \tag{57}$$

where ρ and S are assumed to be the scalar functions and $g := \det g_{ij}$. Since the Fokker-Planck equation is independent of the scalar curvature term the averages of observables depending on x only and their evolution do not depend on the concrete covariant ordering scheme.

Rewriting Eqs. (57) in terms of the density $g^{1/2}\rho$ one can see that the inverse metric is proportional to the diffusion matrix. Besides, from the Heisenberg equations we find a systematic drift,

$$m\frac{d}{dt}\langle x^i\rangle = \left\langle g^{ij}(\partial_j S - A_j) + \frac{\hbar}{2}g^{-1/2}\partial_j(g^{1/2}g^{ij})\right\rangle. \tag{58}$$

Despite that the second term looks noncovariant it behaves under general coordinate transformations like the first term under the assumption that ρ tends to zero at spatial infinity. The transition probability (31) takes the form (cf. [19,45])

$$G(t',x';t,x) = \int \prod_{\tau\in(t,t')} d^d x(\tau) \prod_{\tau\in[t,t')} \frac{d^d p(\tau)}{(2\pi\hbar)^d} \times\exp\left(-\frac{i}{\hbar}\int_t^{t'-d\tau} d\tau\left\{p_i\dot{x}^i + \frac{i}{m}\left[-\frac{g^{ij}p_i p_j}{2} + ip_i\left(g^{ij}(\partial_j S - A_j) + \frac{\hbar}{2}g^{-1/2}\partial_j(g^{1/2}g^{ij})\right)\right]\right\}\right). \tag{59}$$

It is covariant under general coordinate transformations. To prove this by making a change of variables one should take care about the so-called extraterms (see, e.g., [46]) which cancel derivatives of the Jacobian matrices resulting from the noncovariant expression in the exponent.

3. Klein-Kramers equation

Now we turn to another generalization of the Fokker-Planck equation (42), namely, to the Klein-Kramers equation. This equation arises in studying systems of second order stochastic differential equations. Therefore, first, introducing additional variables we reduce such a system to the system of the first order equations of the form (32) and then apply the developed formalism.

Let us consider a system of stochastic equations

$$\dot{x}^i = \upsilon^j, \quad \dot{\upsilon}^j = f^i(t,x,\upsilon) + \nu^j, \tag{60}$$

where ν^j is a Gaussian white noise. In that case the points of the configuration space are the pairs (x,υ). We denote by (p,π) canonically conjugate variables to (x,υ) in the phase space

$$\{x^i, p_j\} = \delta^i_j, \quad \{\upsilon^j, \pi_j\} = \delta^i_j. \tag{61}$$

From the representation (31) it is not difficult to see that this stochastic system is described by the Hamiltonian

$$\hat{H} = \frac{(\hat{\pi}_i - \hat{A}_i)^2}{2} + \hat{\upsilon}^j \hat{p}_i + \hat{A}^0, \tag{62}$$

where $A_\mu(t,x,\upsilon)$ are the gauge fields. The Hamiltonian (62) is merely a general expression [47] at most quadratic in momenta π_i and linear in momenta p_i. By introducing the phase $S(t,x,\upsilon)$ the Schrödinger equations corresponding to the Hamiltonian (62) can be cast into the form

$$\partial_t \rho = -\partial^j_\upsilon \left[-\frac{\hbar}{2}\partial^\upsilon_i \rho + (\partial^\upsilon_i S - A_i)\rho \right] - \partial^j_x(\upsilon_i \rho),$$

$$\partial_t S - A_0 + \frac{(\partial^\upsilon_i S - A_i)^2}{2} + \upsilon^j \partial^x_i S = -\frac{\hbar}{2}\partial^j_\upsilon(\partial^\upsilon_i S - A_i). \tag{63}$$

The first equation is the Klein-Kramers equation, while the second equation is an analog of the Hamilton-Jacobi equation or, from a utilitarian viewpoint, is the definition of A_0. These equations are Lagrangian with the Hamiltonian action of the form (41). Besides, they are invariant with respect to the gauge transformations

$$S(t,x,\upsilon) \to S(t,x,\upsilon) + \xi(t,x,\upsilon),$$

$$A_i(t,x,\upsilon) \to A_i(t,x,\upsilon) + \partial^\upsilon_i \xi(t,x,\upsilon),$$

$$A_0(t,x,\upsilon) \to A_0(t,x,\upsilon) + \dot{\xi}(t,x,\upsilon) + \upsilon^j \partial^x_i \xi(t,x,\upsilon). \tag{64}$$

The covariant derivatives respecting these gauge transformations read as follows:

$$\hat{\mathcal{P}}_0 = \hat{p}_0 + \upsilon^j \hat{p}_i - A_0, \quad \hat{\mathcal{P}}_i = \hat{\pi}_i - A_i. \tag{65}$$

As far as an interpretation of the gauge fields A_i is concerned imposing the unitary gauge $S=0$ we see that these fields are equal to the components of the systematic force f_i with an opposite sign. A path-integral representation of the transition probability (31) reads as

$$G(t',x';t,x) = \int \prod_{\tau\in(t,t')} [d^d x(\tau) d^d \upsilon(\tau)] \prod_{\tau\in[t,t')} \frac{d^d p(\tau) d^d \pi(\tau)}{(2\pi\hbar)^{2d}}$$
$$\times \exp\left\{ -\frac{i}{\hbar}\int_t^{t'-d\tau} d\tau \left[p_i \dot{x}^i + \pi_i \dot{\upsilon}^j + i\left(-\frac{\pi^2}{2} + i\pi^j(\partial^\upsilon_i S - A_i) + i p_i \upsilon^j \right) \right] \right\}. \tag{66}$$

For the Hamiltonians at most quadratic in variables, i.e., for the linear systematic force f^i [48], an evolution of the stochastic system can be easily found from the Heisenberg equations. Here we consider the simplest case,

$$f^i = -\gamma \upsilon^j, \tag{67}$$

just to illustrate how the formalism works. For the force (67) we can choose

$$S = -\gamma\frac{\upsilon^2}{2}, \quad A_i = 0 \Rightarrow A_0 = \gamma^2 \frac{\upsilon^2}{2} - \frac{\hbar}{2}\gamma d, \tag{68}$$

whence in the Heisenberg representation it follows from the definition of the phase S that

$$\langle \varphi | \hat{\pi}_i(0) = -\gamma \langle \varphi | \hat{\upsilon}_i(0), \quad \langle \varphi | \hat{p}_i(0) = 0. \tag{69}$$

A general solution of the Heisenberg equations looks like

$$\hat{p}(t) = \hat{p},$$

$$\hat{x}(t) = \hat{x} + \gamma^{-2}(\hat{p}t - \hat{\pi}) - \gamma^{-1}(\hat{\upsilon} - \gamma^{-2}\hat{p})\sinh \gamma t + \gamma^{-2}\hat{\pi}\cosh \gamma t,$$

$$\hat{\pi}(t) = \hat{\pi}\cosh \gamma t + (\gamma\hat{\upsilon} - \gamma^{-1}\hat{p})\sinh \gamma t,$$

$$\hat{\upsilon}(t) = \gamma^{-2}\hat{p} + (\hat{\upsilon} - \gamma^{-2}\hat{p})\cosh \gamma t + \gamma^{-1}\hat{\pi}\sinh \gamma t, \tag{70}$$

where all the operators at the right-hand side (RHS) of these equations are taken at $t=0$. Making use of Eqs. (69) and (70) and the commutation relations one can obtain an evolution of the average of any observable. For instance,

$$\frac{m}{2}\langle \upsilon^2(t)\rangle = \frac{m\hbar}{4\gamma}d + \left(\frac{m}{2}\langle \upsilon^2\rangle - \frac{m\hbar}{4\gamma}d \right) e^{-2\gamma t}. \tag{71}$$

Then assuming that $\gamma>0$ and the equipartition law is fulfilled we determine the deformation parameter in Eqs. (63),

$$\hbar = \frac{2\gamma kT}{m}. \tag{72}$$

It is interesting to note that the trace of the equipartition law in the Fokker-Planck equation (42) with the Hamiltonian (35) is the relation

$$\lim_{dt\to 0} T\left\{ \frac{m\dot{\hat{x}}^2(t)}{2} dt \right\} = \frac{m}{2\hbar_{FP}}[\hat{x}_i, [\hat{x}^i, \hat{H}]] = \frac{\hbar_{FP}}{2}d, \tag{73}$$

where $\hat{x}(t)$ are the position operators in the Heisenberg representation, T means the chronological ordering, and $\hbar_{FP}$ is the deformation parameter in the Fokker-Planck equation. To reproduce the equipartition law one should formally put $dt=2\gamma^{-1}$.

B. Relativistic particle

In this subsection we consider three relativistic models which under stochastic deformation give rise to relativistic generalizations of the Fokker-Planck and Klein-Kramers equations, and to the equation describing a massive charged particle influenced by external systematic and stochastic forces with the radiation reaction taken into account. In other words, in the last case a dissipation is described by the Lorentz-Dirac force [49,50].

1. Relativistic Fokker-Planck equation

In the previous subsection we saw that the stochastically deformed model of a nonrelativistic particle results in the Fokker-Planck equation. Therefore it is reasonable to expect that stochastic deformation of the model of a relativistic particle gives some relativistic generalization of the Fokker-Planck equation.

The Hamiltonian action of an interacting relativistic particle has the form

$$S_H[x,p,\lambda] = \int d\tau\left[p_\mu \dot{x}^\mu + \frac{\lambda}{2}(\mathcal{P}^2 - m^2)\right], \tag{74}$$

where $\mathcal{P}_\mu := p_\mu - A_\mu$ and A_μ are the gauge fields. The dynamics of the model (74) are governed by one first class constraint. In the proper time parametrization $\lambda = m^{-1}$ the evolution of the position of the particle obeys the equation

$$m\dot{x}^\mu = A^\mu - p^\mu. \tag{75}$$

Thus by analogy with a nonrelativistic model one should expect that the corresponding stochastic equations look in the unitary gauge like

$$m\dot{x}^\mu = A^\mu + \cdots, \tag{76}$$

where dots denote terms vanishing at $\hbar=0$, i.e., in the classical limit A^μ plays the role of the four-momentum of the particle. The problem that one encounters by naive introduction of the noise in the RHS of Eq. (76) is to preserve the reparametrization invariance. In the proper time gauge, for example, we must guarantee the fulfillment of $\dot{x}^2=1$. The stochastic deformation procedure allows us to obtain the Fokker-Planck type equation associated with Eq. (76) respecting the gauge invariance.

We shall deform the model (74) in the gauge of a laboratory time

$$\tau = x^0. \tag{77}$$

A thorough description of quantization of a relativistic particle in the gauge (77) is presented in [51] and we just trace some basic steps of this procedure which are necessary for us. First, we solve the mass-shell constraint with respect to the energy [52]

$$\mathcal{P}_0 - \sqrt{m^2 + (\mathcal{P}_i)^2} = 0. \tag{78}$$

Then we naturally realize the Heisenberg-Weyl algebra of operators in the linear space V of two-component real vectors,

$$\Psi = \begin{bmatrix} \psi'(x) \\ \psi(x) \end{bmatrix} \in V, \quad \hat{x}^\mu := \begin{bmatrix} x^\mu & 0 \\ 0 & x^\mu \end{bmatrix},$$

$$\hat{p}_\mu := \begin{bmatrix} -\hbar\partial_\mu & 0 \\ 0 & -\hbar\partial_\mu \end{bmatrix}. \tag{79}$$

Define the dual linear space V^* as the space of real linear functionals acting on elements of V by the rule

$$\langle\Phi|\Psi\rangle := -\int d\mathbf{x}\Phi^+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Psi = -\int d\mathbf{x}[\varphi\psi' + \varphi'\psi], \tag{80}$$

where

$$\Phi = \begin{bmatrix} \varphi'(x) \\ \varphi(x) \end{bmatrix}. \tag{81}$$

The pure states are as usual the projectors of the form (23) [53].

The above construction allows us to realize the constraint (78) without going to pseudodifferential operators:

$$\hat{T} := \hat{\mathcal{P}}_0 - \begin{bmatrix} 0 & m^2 + (A_i + \hbar\partial_i)^2 \\ 1 & 0 \end{bmatrix}. \tag{82}$$

The dynamics of a pure state of the stochastic system are governed in the gauge (77) by the two Schrödinger equations

$$\hat{T}|\Psi\rangle = \langle\Phi|\hat{T} = 0. \tag{83}$$

In quantum mechanics they are simply the Klein-Gordon equation in the first order formalism [54]. In our case these equations are equivalent to

$$(\hat{\mathcal{P}}^2 - m^2)\psi = 0, \quad \psi' = \hat{\mathcal{P}}_0\psi, \quad (\hat{\mathcal{P}}^{+2} - m^2)\varphi = 0, \quad \varphi' = \hat{\mathcal{P}}_0^+\varphi, \tag{84}$$

where the cross denotes a formal conjugation. When the gauge fields vanish these equations are the Klein-Gordon equations for tachyons [55]. On the solutions of the Schrödinger equations (83) the inner product (80) reduces to

$$\langle\Phi|\Psi\rangle = -\int d\mathbf{x}[\varphi\hat{\mathcal{P}}_0\psi + \psi\hat{\mathcal{P}}_0^+\varphi]. \tag{85}$$

The probability density function corresponding to the state $|\Phi\rangle\langle\Psi|$ is proportional to the integrand of Eq. (85).

The Heisenberg equations associated with the Schrödinger equations (83) allow us to obtain the evolution of an average of any physical observable and, in particular, to establish a stochastic interpretation of the gauge fields A_μ. However, we shall act in a different way to derive a direct stochastic interpretation of Eqs. (84) in terms of the Langevin equation of the form (32). To this aim we first represent the fields $\varphi(x)$ and $\psi(x)$ as

$$\varphi(x) = e^{1/\hbar S(x)}, \quad \psi(x) = \tilde{\rho}(x) e^{-1/\hbar S(x)}, \tag{86}$$

substitute them into Eqs. (84), and fix the unitary gauge $S=0$. Then we have [56]

$$\partial^\mu \left[\frac{\hbar}{2} \partial_\mu \tilde{\rho} + A_\mu \tilde{\rho} \right] = 0, \quad A^2 - m^2 = \hbar \partial^\mu A_\mu. \tag{87}$$

The first equation is the conservation law of the probability density current

$$j_\mu = \frac{1}{m} \left(A_\mu + \frac{\hbar}{2} \partial_\mu \right) \tilde{\rho}, \tag{88}$$

while the second equation is the deformed mass-shell condition or the mere definition of $A^0(x)$. For given $A_i(x)$ we take the solution of the second equation in Eq. (87) which is regular in $\hbar$ and possesses the following classical limit:

$$A_0|_{\hbar=0} = \sqrt{m^2 + (A_i)^2}. \tag{89}$$

The solution defined in this way is unique. Consequently, if one has $j_\mu(x)$ at the initial moment then one can find from Eqs. (88) the function $\tilde{\rho}(x)$ and its first derivative with respect to time at this initial moment and solve the Cauchy problem.

For the potential and stationary relativistic flow with a constant specific relativistic enthalpy [37] entraining the Brownian particle we can choose $A_i = \partial_i U(\mathbf{x})$, where U is a Lorentz scalar. Then the stationary probability distribution function following from Eq. (88) looks like

$$j^0 = \sqrt{1 + (A_i)^2/m^2} \exp(-2U/\hbar)/Z = \gamma \exp(-2U/\hbar)/Z,$$

$$Z = \text{const}, \tag{90}$$

where γ is the Lorentz factor. The power of the γ factor is uniquely specified by the requirement that j^0 is the zeroth component of the four-vector.

It is easy to see from Eq. (88) that the derived Eqs. (84) lead to negative probabilities when the probability density function changes rapidly on the spatial (time) scales less than or comparable with the "Compton wave length" $\hbar m^{-1}$. This is in a perfect analogy with the well-known property of the Klein-Gordon equation in relativistic quantum mechanics [57]. Therefore Eqs. (84) have a correct stochastic interpretation only for the fields with a characteristic scale of variations much larger than $\hbar m^{-1}$.

Despite the above remark we consider an evolution of the probability density function localized in one point $x^i = y^i$ at the initial moment $x^0 = y^0$:

$$j^0(y^0, x^i) = \delta^{d-1}(x^i - y^i). \tag{91}$$

The solution of the Cauchy problem to the first equation in Eq. (87) is obtained via a convolution of the retarded Green function with the initial distribution

$$\left(1 + \frac{1}{2} a \hat{p}_0 \right) \delta^d(x - y), \tag{92}$$

where the function $a(x)$ is the regular in $\hbar$ solution of the equation

$$\left(A_0 + \frac{\hbar}{2} \partial_0 \right) a(x) = 1. \tag{93}$$

Thus, formally, the transition probability is

$$G(x,y) = \left(A_0 - \frac{1}{2} \hat{p}_0 \right) \frac{1}{\frac{1}{2}\hat{p}^2 - \hat{p}^\mu A_\mu} \left(1 + \frac{1}{2} a \hat{p}_0 \right) \delta^d(x - y). \tag{94}$$

The transition probability (94) does not possess the defining property of the Markov process and it cannot be straightforwardly represented by a path integral. To overcome this difficulty we represent the transition probability (94) as the sum of the transition probabilities possessing the Markov property with respect to some new variable (fictitious time). The stated procedure is an analog of what is called in relativistic quantum mechanics the Fock's fifth parameter (proper time) formalism [58,59].

Under some technical assumptions on the transition probability (94) the following equality for the operator $\hat{G}$ with the kernel (94) holds:

$$\hat{G} = \hbar^{-1} \int_0^\infty d\tau e^{-\tau/\hbar \hat{H}},$$

$$\hat{H} := \left(1 + \frac{1}{2} a \hat{p}_0 \right)^{-1} \left(\frac{1}{2} \hat{p}^2 - \hat{p}^\mu A_\mu \right) \left(A_0 - \frac{1}{2} \hat{p}_0 \right)^{-1}. \tag{95}$$

This formula is understood in the distributional sense. It is derived by an application of the descent method (see, e.g., [21,22]) to the equation

$$-\hbar \partial_\tau \psi(\tau, x) = \hat{H} \psi(\tau, x), \quad \psi(0, x) = \delta^d(x - y), \tag{96}$$

and is valid when the integral (95) converges. It is not difficult to prove the convergence of this integral in the case of the constant fields $A_i(x)$. In a general case the integral (95) can be defined at least perturbatively.

The evolution operator $\exp[-\hbar^{-1} \tau \hat{H}]$ respects the Markov property and conserves the probability density function normalization. Its kernel in the x representation can be interpreted as the probability of particle's arrival to the point x of the space-time from the point y with the value of the fictitious time τ. The sum over all such particles with different fictitious times gives the total observable transition probability (94).

Now we represent the kernel of the evolution operator $\exp[-\hbar^{-1} \tau \hat{H}]$ by a path integral along lines expounded in Sec. II and introduce the corresponding Langevin equation (32). It seems impossible to find an explicit simple expression for the probability density function of noise for arbitrary fields $A_i(x)$ but some properties of this distribution are easily derived.

First, in the classical limit $\hbar \to 0$ the Hamiltonian (95) reduces to

$$\hat{H} = -\hat{p}^\mu A_\mu / A_0, \tag{97}$$

where A_0 is specified by Eq. (89). Consequently, the probability density function of noise reads

$$F(\nu,\tau,x)\underset{\hbar\to 0}{\to}\delta(\nu^0-1)\delta^{d-1}(\nu^i-A^i/\sqrt{m^2+(A_i)^2}). \quad (98)$$

We see that in the classical limit the fictitious time turns into the physical time x^0 of the laboratory frame. Besides, the above formula gives the interpretation to the fields $A_i(x)$, which confirms our expectations.

Second, in the nonrelativistic limit $A_0\approx m$ and $m^{-1}\|\hat{p}_0\|\ll 1$, where $\hat{p}_0$ is assumed to act on the probability density function, the Hamiltonian (95)is rewritten as

$$\hat{H}=-\frac{(\hat{p}_i)^2}{2m}-\frac{\hat{p}^\mu A_\mu}{m}, \quad (99)$$

whence the standard Gaussian distribution for the noise follows. As in the classical limit the fictitious time τ is the physical time x^0.

Thus Eqs. (84) generalize the Fokker-Planck equation (42) in the same sense as the Klein-Gordon equation generalizes the Schrödinger equation. In the classical limit these equations describe the same physical system and in the nonrelativistic limit Eqs. (84) reduce to the Fokker-Planck equation (42). However, it is worthwhile to note that the above interpretation in terms of the fictitious time has some weaknesses. Namely, since we start from the probability density function (91) localized in one point the obtained equations are ill-defined. This inevitably leads to negative probabilities although they disappear in the classical and nonrelativistic limits, when $\hbar m^{-1}$ tends to zero. Moreover, the negative probabilities appear to result in the probability density function of the noise acting on the particle arrived to the point x with the fictitious time τ being not equal to zero in the superluminal region. But, of course, the sum over all the particles with different τ gives rise to the transition probability with the support bounded by the light cone. A detailed investigation of these peculiarities will be given elsewhere.

2. Relativistic Klein-Kramers equation

To formulate a relativistic generalization of the Klein-Kramers equation we start from a classical relativistic system with the Hamiltonian action of the form [cf. Eq. (62)]

$$S_H[x,p,y,\pi,\lambda,\mu]=\int d\tau(p_\mu\dot{x}^\mu+\pi_\mu\dot{y}^\mu-\lambda T_1-\mu T_2),$$

$$T_1:=\frac{1}{2}g^{\mu\nu}(\pi_\mu+A_\mu)(\pi_\nu+A_\nu)+y^\mu p_\mu+\phi, \quad T_2:=y^2-m^2, \quad (100)$$

where dots denote the derivatives with respect to the parameter τ, m is a mass of the particle, $A_\mu(x,y)$ and $\phi(x,y)$ are gauge fields, $g^{\mu\nu}(x,y)$ is a symmetric Lorentz tensor orthogonal to y_μ. In other words the dynamics of the system are governed by two first class constraints. The first constraint generates reparametrizations of the world line of the particle, the second constraint being just the mass-shell condition. These two constraints are in the Abelian involution by virtue of the property of the tensor $g^{\mu\nu}$.

A part of the equations of motion in what we are interested in reads as follows:

$$\dot{x}^\mu=\lambda y^\mu, \quad \dot{y}^\mu=\lambda g^{\mu\nu}(A_\nu+\pi_\nu), \quad y^2=m^2. \quad (101)$$

By analogy with a nonrelativistic case we expect that in the classical limit $\hbar\to 0$ and in the unitary gauge $S=0$ the stochastically deformed system (100) will describe a relativistic particle with the equations of motion

$$\dot{y}^\mu=m^{-1}g^{\mu\nu}(x,y)A_\nu(x,y)\sqrt{\dot{x}^2}, \quad m\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}}=y^\mu. \quad (102)$$

Hence in the unitary gauge and in the classical limit $m^{-1}g^{\mu\nu}A_\nu$ is expected to be a systematic force acting on the particle. As in the previously considered model a naive introduction of the noise to the RHS of the first equation in Eq. (102) spoils the reparametrization invariance. To preserve it we have to demand from the noise correlators

$$y^\mu(\tau)\langle\nu_\mu(\tau)\nu_{\mu_1}(\tau_1)\cdots\nu_{\mu_n}(\tau_n)\rangle=0, \quad (103)$$

where $\nu_\mu(\tau)$ is the noise. The identities (103) can be viewed as the Ward-Takahashi identities for this model (see a detailed discussion in [60,61]). By the use of the stochastic deformation procedure we shall find below the general form of the Fokker-Planck equation associated with Eq. (102) respecting the reparametrization invariance in the case of the Gaussian white noise. A generalization of the obtained equation to the case of an arbitrary noise distribution law will be obvious.

To deform the model (100) we naturally realize the Heisenberg-Weyl algebra in the linear space V of smooth functions $\psi(x,y)$:

$$\hat{x}^\mu=x^\mu, \quad \hat{p}_\mu=-\hbar\partial^x_\mu, \quad \hat{y}^\mu=y^\mu, \quad \hat{\pi}_\mu=-\hbar\partial^y_\mu. \quad (104)$$

Then the constraints (100) are the Weyl-ordered operators

$$\hat{T}_1=\frac{1}{8}(\hat{\Pi}_\mu\hat{\Pi}_\nu g^{\mu\nu}+2\hat{\Pi}_\mu g^{\mu\nu}\hat{\Pi}_\nu+g^{\mu\nu}\hat{\Pi}_\mu\hat{\Pi}_\nu)+y^\mu\hat{p}_\mu+\phi,$$

$$\hat{T}_2=y^2-m^2, \quad (105)$$

where $\hat{\Pi}_\mu=\hat{\pi}_\mu+A_\mu$. These constraints are in the Abelian involution as well [62]. The physical pure states of the system are singled out by the conditions

$$y_0^{-1}\hat{T}_1|\psi\rangle=\langle\varphi|y_0^{-1}\hat{T}_1=0, \quad \hat{T}_2|\psi\rangle=\langle\varphi|\hat{T}_2=0, \quad (106)$$

where the inner product is defined in a standard way and we resolve the first constraint with respect to $\hat{p}_0$. Equations (106) are invariant under the gauge transformations

$$\varphi\to e^{\xi/\hbar}\varphi, \quad \psi\to e^{-\xi/\hbar}\psi,$$

$$A_\mu\to A_\mu-\partial^y_\mu\xi, \quad \phi\to\phi-y^\mu\partial^x_\mu\xi. \quad (107)$$

Besides the transformation of the gauge fields A_μ of the form

$$A_\mu(x,y)\to A_\mu(x,y)+y_\mu\zeta(x,y), \quad (108)$$

leaves Eqs. (106) unchanged. Therefore the fields A_μ have $d-1$ independent components and in the massive case one can make them orthogonal to y^μ. Introducing the phase

$$\varphi(x,y)=e^{1/\hbar S(x,y)},\quad \psi(x,y)=y^0\tilde{\rho}(x,y)e^{-1/\hbar S(x,y)},\quad (109)$$

we reduce the first pair of equations (the Schrödinger equations) in formula (106) to the system

$$\partial^y_\mu\left[\frac{\hbar}{2}g^{\mu\nu}\partial^y_\nu\tilde{\rho}-g^{\mu\nu}(\partial^y_\nu S+A_\nu)\tilde{\rho}\right]-\partial^x_\mu(y^\mu\tilde{\rho})=0,$$

$$\frac{1}{2}g^{\mu\nu}(\partial^y_\mu S+A_\mu)(\partial^y_\nu S+A_\nu)+y^\mu\partial^x_\mu S+\phi$$

$$=-\frac{\hbar}{2}\partial^y_\mu[g^{\mu\nu}(\partial^y_\nu S+A_\nu)]-\frac{\hbar^2}{8}\partial^y_{\mu\nu}g^{\mu\nu}. \quad (110)$$

The first equation is the conservation law of the probability density current whose time component is the probability density function. The second equation is the definition of ϕ.

The second pair of equations in Eq. (106) is taken into account by the equalities

$$\tilde{\rho}=\delta(y^2-m^2)\bar{\rho}(x,y^i),\quad S=\ln\,\delta(y^2-m^2)+\bar{S}(x,y^i). \quad (111)$$

On substituting these equalities to the first equation in Eq. (110) we obtain a relativistic generalization of the Klein-Kramers equation [23–28,63,64]

$$\partial^y_i\left[\frac{g^{ij}}{y^0}\left(\frac{\hbar}{2}\partial^y_j\bar{\rho}-(\partial^y_j\bar{S}+A_j)\bar{\rho}\right)\right]-\partial^x_\mu\left(\bar{\rho}\frac{y^\mu}{y^0}\right)=0,\quad (112)$$

where the positive root of the mass-shell condition is chosen: $y^0=\sqrt{m^2+(y_i)^2}$. The logarithm of the δ function in Eq. (111) does not actually contribute to Eqs. (112) because of the orthogonality of the tensor $g^{\mu\nu}$ to y_μ. Now it is not difficult to construct a path-integral representation of the transition probability and the corresponding Langevin equations (32). For brevity we give the Langevin equations only,

$$\dot{x}^\mu=\frac{y^\mu}{y^0},\quad \dot{y}^i=\frac{g^{ij}}{y^0}(\partial^y_j\bar{S}+A_j)+\frac{\hbar}{2}\partial^y_j\left(\frac{g^{ij}}{y^0}\right)+\nu^i,$$

$$\langle\nu^i(\tau)\rangle=0,\quad \langle\nu^i(\tau)\nu^j(\tau')\rangle=\hbar\frac{g^{ij}}{y^0}\delta(\tau-\tau'),\quad (113)$$

where ν^i is a Gaussian white noise and we assume that $g^{ij}(x,y)$ is positively definite. Recall that the stochastic equations (113) are understood in the Ito sense. Thus we confirm our expectations that Eqs. (112) and (113) describe a relativistic particle in the gauge of a laboratory time (77) under the influence of noise.

As far as the tensor $g^{\mu\nu}$ is concerned its physical meaning is comprehended from the Langevin equations (113). We remark only two possible choices:

$$g_1^{\mu\nu}=\frac{y^\mu y^\nu}{y^2}-\eta^{\mu\nu},$$

$$g_2^{\mu\nu}=-\left(\eta^\mu_\rho-\frac{n^\mu y_\rho}{n_\sigma y^\sigma}\right)\left(\eta^{\rho\nu}-\frac{y^\rho n^\nu}{n_\lambda y^\lambda}\right),\quad n^2=1.\quad (114)$$

The first tensor implies that the effect of the stochastic force on a relativistic particle is isotropic in a momentary comoving frame. The second tensor corresponds to the isotropic influence of the stochastic force in the frame distinguished by the vector n^μ. It is interesting to note that in the first case, which is minimal from the mathematical point of view since new objects are not introduced, the tensor g_1^{ij} can be interpreted as the inverse metric in the momentum space with the coordinates y^i. Keeping in mind that

$$(\det g_1^{ij})^{1/2}=m^{-1}y^0,\quad (115)$$

we can rewrite Eq. (112) in a covariant way. The metric g^1_{ij} is a metric of a constant positive curvature and, consequently, it is conformally flat. The second tensor g_2^{ij} becomes Euclidean provided that $n^\mu=\delta^\mu_0$.

To conclude this example we briefly discuss a representation of the transition probability in terms of the proper time. The main observation allowing us to formulate Eqs. (113) in the proper time gauge is the following. The retarded Green function of the twisted forward Schrödinger equation (106), that is, the transition probability, is the kernel of

$$-\frac{1}{\varphi y_0^{-1}\hat{T}_1\varphi^{-1}}=-y^0\frac{1}{\tilde{\varphi}\hat{T}_1\tilde{\varphi}^{-1}}=\int_0^\infty d\tau\frac{y^0}{\hbar m}e^{\tau/\hbar m\tilde{\varphi}\hat{T}_1\tilde{\varphi}^{-1}},\quad (116)$$

where $\tilde{\varphi}:=y_0^{-1}\varphi$ and the last equality is proved by the descent method as was done in considering a relativistic generalization of the Fokker-Planck equation. The operator in the exponent is proportional to the operator acting on $\tilde{\rho}$ in the first equation in formula (110).

The kernel

$$\langle x,y|\exp\left[\frac{\tau}{\hbar m}\tilde{\varphi}\hat{T}_1\tilde{\varphi}^{-1}\right]|x',y'\rangle \quad (117)$$

is the probability density of particle's arrival to the point (x,y) from the point (x',y') with the proper time τ measured in the momentary comoving frame. The factor y^0m^{-1} in Eq. (116) is caused by the passage from the momentary comoving frame to the laboratory frame. The transition probability possesses the Markov property with respect to the proper time τ. It conserves the normalization of the probability density function, i.e., the probability to find a particle with the proper time τ in some point of the space-time with a certain momentum is equal to unity. Since the generator of evolution commutes with the operator $\hat{T}_2$ the transition probability (117) respects the mass-shell condition.

The Langevin equations associated with Eq. (117) read as

$$m\dot{x}^\mu=y^\mu,\quad \dot{y}^\mu=m^{-1}g^{\mu\nu}(\partial^y_\nu S+A_\nu)+\hbar m^{-1}\partial^y_\nu g^{\mu\nu}/2+\nu^\mu,$$

$$\langle\nu^\mu(\tau)\rangle=0,\quad \langle\nu^\mu(\tau)\nu^\nu(\tau')\rangle=\hbar m^{-1}g^{\mu\nu}\delta(\tau-\tau'),\quad (118)$$

whence we infer that τ is indeed the proper time. As before the noise ν^μ is a Gaussian white noise. The representation (116) of the transition probability is especially useful when a relativistic particle has a finite lifetime. In that case the integral (116) over τ is cut on the upper limit by the lifetime of the particle provided, of course, that the particle was created at the initial moment.

3. Stochastic Lorentz-Dirac equation

A relativistic generalization of the Klein-Kramers equation regarded in the preceding example does not apply to a charged particle with a radiation reaction taken into account. If we apply the above scheme to the so-called Landau-Lifshitz equation [65], which is obtained from the Lorentz-Dirac equation by the reduction of order procedure, then the vanishing external systematic force entails a zero dissipation force in the resulting stochastic equation. That is why we should stochastically deform the Lorentz-Dirac equation itself. The procedure of stochastic deformation is very similar to what we have considered in the previous example. Therefore we mention the crucial points only.

We start from the Hamiltonian action of the form

$$S_H[x,\bar{x},y,\bar{y},w,\bar{w},\lambda,\mu,\varkappa]=\int d\tau(\bar{x}_\mu\dot{x}^\mu+\bar{y}_\mu\dot{y}^\mu+\bar{w}_\mu\dot{w}^\mu-\lambda T_1-\mu T_2-\varkappa T_3),$$

$$T_1:=\frac{1}{2}g^{\mu\nu}\bar{W}_\mu\bar{W}_\nu-w^2y^\mu\bar{W}_\mu+w^\mu\bar{y}_\mu+y^\mu\bar{x}_\mu+\phi,$$

$$T_2:=y^2-1,\quad T_3:=yw, \tag{119}$$

where $\bar{W}_\mu:=\bar{w}_\mu+A_\mu$ is a covariant momentum, $A_\mu(x,y,w)$ and $\phi(x,y,w)$ are gauge fields, and $g^{\mu\nu}(x,y,w)$ is a symmetric Lorentz tensor orthogonal to y_μ. For simplicity we assume that the particle has a unit mass. The constraints T_1, T_2, and T_3 are of the first class with the algebra

$$\{T_1,T_2\}=-2T_3,\quad \{T_1,T_3\}=w^2T_2. \tag{120}$$

A relevant part of the equations of motion resulting from the action (119) reads as

$$\dot{x}^\mu=\lambda y^\mu,\quad \dot{y}^\mu=\lambda w^\mu,\quad \dot{w}^\mu=\lambda(g^{\mu\nu}\bar{W}_\nu-w^2y^\mu),$$

$$y^2=1,\quad yw=0. \tag{121}$$

These equations suggest that in the classical limit and in the unitary gauge the stochastically deformed model (119) describes a relativistic particle obeying the equations of motion

$$\dddot{x}^\mu+\ddot{x}^2\dot{x}^\mu=g^{\mu\nu}A_\nu,\quad \dot{x}^\mu=y^\mu,\quad \dot{y}^\mu=w^\mu, \tag{122}$$

where the derivatives are taken with respect to the proper time. The LHS of the first equation is proportional to the Lorentz-Dirac force and we refer to this equation as the Lorentz-Dirac equation. The mass term is contained in the RHS of this equation.

The Heisenberg-Weyl algebra is naturally realized in the linear space V of smooth functions $\psi(x,y,w)$:

$$\hat{x}^\mu=x^\mu,\quad \hat{\bar{x}}_\mu=-\hbar\partial^x_\mu,\quad \hat{y}^\mu=y^\mu,\quad \hat{\bar{y}}_\mu=-\hbar\partial^y_\mu,$$

$$\hat{w}^\mu=w^\mu,\quad \hat{\bar{w}}_\mu=-\hbar\partial^w_\mu. \tag{123}$$

The constraints turn into the appropriate Weyl-ordered operators. Their algebra coincides with the classical algebra of constraints (120). The physical states of the stochastic system are specified by

$$y_0^{-1}\hat{T}_1|\psi\rangle=\langle\varphi|y_0^{-1}\hat{T}_1=0,$$

$$\hat{T}_2|\psi\rangle=\langle\varphi|\hat{T}_2=0,\quad \hat{T}_3|\psi\rangle=\langle\varphi|\hat{T}_3=0, \tag{124}$$

where the standard inner product is understood. The first pair of equations in formula (124) is just the forward and backward Schrödinger equations.

Making a substitution of the form (109) into the first equation in Eq. (124) we arrive at [66]

$$\partial^w_\mu\left(\frac{\hbar}{2}g^{\mu\nu}\partial^w_\nu\tilde{\rho}-[g^{\mu\nu}(\partial^w_\nu S+A_\nu)-w^2y^\mu]\tilde{\rho}\right)-\partial^y_\mu(w^\mu\tilde{\rho})-\partial^x_\mu(y^\mu\tilde{\rho})=0,$$

$$\frac{1}{2}g^{\mu\nu}(\partial^w_\mu S+A_\mu)(\partial^w_\nu S+A_\nu)-w^2y^\mu\partial^w_\mu S+w^\mu\partial^y_\mu S+y^\mu\partial^x_\mu S-yw+\phi=-\frac{\hbar}{2}\partial^w_\mu[g^{\mu\nu}(\partial^w_\nu S+A_\nu)]-\frac{\hbar^2}{8}\partial^w_{\mu\nu}g^{\mu\nu}. \tag{125}$$

As usual the first equation is the conservation law of the probability density current. The probability density function is the time component of this current. The rest of the equations defining the physical state results in

$$\tilde{\rho}=\delta(y^2-1)\delta(yw)\bar{\rho}(x,y^i,w^i),$$

$$S=\ln[\delta(y^2-1)\delta(yw)]+\bar{S}(x,y^i,w^i). \tag{126}$$

Under the above conditions the first equation in Eq. (125) can be reduced after a little algebra to

$$\partial^w_i\left[\left(\frac{\hbar}{2}g^{ij}\partial^w_j-[g^{ij}(\partial^w_j\bar{S}+A_j)-w^2y^i]\right)\frac{\bar{\rho}}{y_0^2}\right]-\partial^y_i\left(w^i\frac{\bar{\rho}}{y_0^2}\right)-\partial^x_\mu\left(y^\mu\frac{\bar{\rho}}{y_0^2}\right)=0, \tag{127}$$

where we have assigned $y^0=\sqrt{1+(y_i)^2}$ and $w^0=y_0^{-1}y_iw_i$. Recall that the probability density function is $y_0^{-1}\bar{\rho}$. Now a path-integral representation of the transition probability associated with Eq. (127) is straightforward. The Langevin equations (32) with the Ito prescription become

$$\dot{x}^\mu=\frac{y^\mu}{y^0},\quad \dot{y}^i=\frac{w^i}{y^0},$$

$$\dot{w}^i=\frac{g^{ij}}{y^0}(\partial^w_j\bar{S}+A_j)-w^2\frac{y^i}{y^0}+\frac{\hbar}{2}\partial^w_j\left(\frac{g^{ij}}{y^0}\right)+\nu^i,$$

$$\langle\nu^i(\tau)\rangle=0,\quad \langle\nu^i(\tau)\nu^j(\tau')\rangle=\hbar\frac{g^{ij}}{y^0}\delta(\tau-\tau'), \tag{128}$$

where ν^i is a Gaussian white noise and $g^{ij}(x,y,w)$ is positively definite. As regards explicit forms of the tensor g^{ij}, see the discussion in the previous example [67]. The obtained stochastic equations are equivalent to the Lorentz-Dirac Eq. (122) with a random force which is rewritten in the gauge of a laboratory time, that is, Eq. (128) is the stochastic Lorentz-Dirac equation.

The proper time representation of the transition probability from the point (x',y',w') to the point (x,y,w) takes the form

$$\langle x,y,w|\int_0^\infty d\tau\hbar^{-1}y^0\exp\left[\frac{\tau}{\hbar}\tilde{\varphi}\hat{T}_1\tilde{\varphi}^{-1}\right]|x',y',w'\rangle, \quad (129)$$

where $\tilde{\varphi}:=y_0^{-1}\varphi$ and $\tilde{\varphi}\hat{T}_1\tilde{\varphi}^{-1}$ is the operator acting on $\tilde{\rho}$ in the first equation of Eq. (125). Besides, the initial state should satisfy the mass-shell condition and its differential consequence. Therefore the proper time stochastic Lorentz-Dirac equation is the following:

$$\dot{x}^\mu = y^\mu, \quad \dot{y}^\mu = w^\mu,$$

$$\dot{w}^\mu = g^{\mu\nu}(\partial_\nu^w S + A_\nu) - w^2 y^\mu + \frac{\hbar}{2}\partial_\nu^w g^{\mu\nu} + \nu^\mu,$$

$$\langle \nu^\mu(\tau)\rangle = 0, \quad \langle \nu^\mu(\tau)\nu^\nu(\tau')\rangle = \hbar g^{\mu\nu}\delta(\tau-\tau'), \quad (130)$$

where ν^μ is a Gaussian white noise.

C. Free relativistic field models

In this subsection we consider the stochastic deformation of two free models: a relativistic real scalar field and an electromagnetic field. We regard the scalar field model as a prototype of any field model. Besides, it is the simplest model of a one-dimensional crystal in the continuum limit, where the scalar field describes the displacements of points of a medium from their equilibrium positions. The stochastically deformed model of Maxwell's fields allows us to touch the problem of electromagnetic fluctuations [29] bringing it in accordance with the general scheme advocated in this paper.

1. Scalar field

Consider a real scalar field $\phi(x)$ on the Minkowski space $\mathbb{R}^{1,d-1}$ with the dynamics governed by the action

$$S[\phi] = \frac{1}{2}\int d^dx(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2), \quad (131)$$

where m is a mass. The corresponding Hamiltonian action has the form

$$S_H[\phi,\pi] = \int d^dx\left[\pi\dot{\phi} - \frac{1}{2}(\pi^2 + \partial_i\phi\partial_i\phi + m^2\phi^2)\right], \quad (132)$$

where the dot denotes the derivative with respect to time. The Hamilton equations are

$$\dot{\phi} = \pi, \quad \dot{\pi} = \Delta\phi - m^2\phi. \quad (133)$$

The Noether theorem gives the conserved four-momentum

$$\mathcal{P}_0 \equiv H = \frac{1}{2}\int d\mathbf{x}(\pi^2 + \partial_i\phi\partial_i\phi + m^2\phi^2), \quad \mathcal{P}_i = \int d\mathbf{x}\,\pi\partial_i\phi, \quad (134)$$

which is the generator of translations.

Now we are going to define the stochastic deformation. The canonical coordinates $\phi(\mathbf{x})$ and canonical momenta $\pi(\mathbf{x})$ turn into the generators of the Heisenberg-Weyl algebra,

$$[\hat{\phi}(\mathbf{x}),\hat{\pi}(\mathbf{y})] = \hbar\delta(\mathbf{x}-\mathbf{y}). \quad (135)$$

Since we deform the linear model its Heisenberg equations coincide with the Hamilton equations. The Heisenberg equations can be formally solved by means of the expansion in terms of the complete set of solutions of the classical equations of motion in the form of plane waves. The field operator and its canonical conjugate read as

$$\hat{\phi}(x) = \int\frac{d\mathbf{p}}{(2\pi)^{d-1}}\frac{1}{\sqrt{p_0}}[\hat{a}(\mathbf{p})\cos(p_\mu x^\mu) + \hat{b}(\mathbf{p})\sin(p_\mu x^\mu)],$$

$$\hat{\pi}(x) = \int\frac{d\mathbf{p}}{(2\pi)^{d-1}}\sqrt{p_0}[\hat{b}(\mathbf{p})\cos(p_\mu x^\mu) - \hat{a}(\mathbf{p})\sin(p_\mu x^\mu)], \quad (136)$$

where $p_0:=\sqrt{m^2+\mathbf{p}^2}$. As it follows from Eq. (135) the operators $\hat{a}(\mathbf{p})$ and $\hat{b}(\mathbf{p})$ obey the commutation relations

$$[\hat{a}(\mathbf{p}),\hat{b}(\mathbf{k})] = \hbar(2\pi)^{d-1}\delta(\mathbf{p}-\mathbf{k}). \quad (137)$$

The Weyl-ordered operators of the generators of translations (134) look like

$$\hat{\mathcal{P}}_\mu = \int\frac{d\mathbf{p}}{(2\pi)^{d-1}}\frac{p_\mu}{2}[\hat{a}^2(\mathbf{p}) + \hat{b}^2(\mathbf{p})]. \quad (138)$$

The solution (136) also implies that

$$[\hat{\phi}(x),\hat{\phi}(y)] = \hbar\int\frac{d\mathbf{p}\,\sin[p_\mu(y-x)^\mu]}{(2\pi)^{d-1}p_0} = -2\hbar\,\mathrm{sgn}(x^0-y^0)\bar{G}(x-y), \quad (139)$$

where $\bar{G}(x-y)$ is the symmetric Green function (see, e.g., [68]),

$$(\Box + m^2)\bar{G}(x) = \delta^d(x), \quad \bar{G}(x-y) = \bar{G}(y-x). \quad (140)$$

As in the case of quantum field theory two field operators separated by a spacelike interval commute.

The key ingredient of stochastic field theory is the propagator, which we define in a standard way,

$$G(x,y):=\langle T\{\hat{\phi}(x)\hat{\phi}(y)\}\rangle, \quad (141)$$

where T is a chronological ordering. Any other correlator can be expressed in terms of the propagator by means of the Wick theorem and the commutation relation (139).

Certainly, an explicit form of the propagator depends on the state entering in its definition. Nevertheless, the propagator possesses the properties following from the rule of differentiation of the chronological ordering and the Heisenberg equation,

$$(\Box_x + m^2)G(x,y) = -\hbar\delta^d(x-y), \quad G(x,y) = G(y,x), \tag{142}$$

regardless of the state which does not depend on time explicitly. Therefore the propagator can be represented as

$$G(x,y) = -\hbar\bar{G}(x-y) + \cdots, \tag{143}$$

where dots denote some solution of the Klein-Gordon equation.

Moreover, if the state $\hat{\rho}$ is translation invariant,

$$[\hat{\rho},\hat{\mathcal{P}}_\mu] = 0, \tag{144}$$

then the obvious identity

$$\mathrm{Tr}([\hat{\mathcal{P}}_\mu,\hat{\rho}T\{\hat{\phi}(x)\hat{\phi}(y)\}]) = 0 \tag{145}$$

entails a translational invariance of the propagator, viz.

$$G(x,y) = G(x-y). \tag{146}$$

Analogously it can be proved that an invariance of the state with respect to the Lorentz transformations results in a dependence of the propagator on the space-time interval only.

Unlike quantum field theory there is no distinguished ground state in our case since the Hamiltonian (138) has an unbounded spectrum. That is why we define the state $\hat{\rho}$ in such a way that the propagator (141) becomes minimal in view of the above considerations, that is, the dotted terms in formula (143) vanish.

Let us achieve this goal gradually. First, suppose that $\hat{\rho}$ commutes with the Hamiltonian, i.e., the state is invariant with respect to translations in time. Then from the equality

$$\langle[\hat{a}(\mathbf{p})\hat{b}(\mathbf{k}),\hat{H}]\rangle = p_0\langle\hat{b}(\mathbf{p})\hat{b}(\mathbf{k})\rangle - k_0\langle\hat{a}(\mathbf{p})\hat{a}(\mathbf{k})\rangle = 0, \tag{147}$$

and the same equality with $\mathbf{p}$ and $\mathbf{k}$ interchanged we find

$$\langle\hat{a}(\mathbf{p})\hat{a}(\mathbf{k})\rangle = \langle\hat{b}(\mathbf{p})\hat{b}(\mathbf{k})\rangle = f(\mathbf{p},\mathbf{k})\delta(p_0-k_0),$$
$$f(\mathbf{p},\mathbf{k}) = f(\mathbf{k},\mathbf{p}). \tag{148}$$

Furthermore, from the equalities

$$\langle[\hat{a}(\mathbf{p})\hat{a}(\mathbf{k}),\hat{H}]\rangle = \frac{1}{2}\langle k_0[\hat{a}(\mathbf{p})\hat{b}(\mathbf{k}) + \hat{b}(\mathbf{k})\hat{a}(\mathbf{p})] + p_0[\hat{a}(\mathbf{k})\hat{b}(\mathbf{p}) + \hat{b}(\mathbf{p})\hat{a}(\mathbf{k})]\rangle = 0,$$

$$\langle[\hat{b}(\mathbf{p})\hat{b}(\mathbf{k}),\hat{H}]\rangle = -\frac{1}{2}\langle k_0[\hat{b}(\mathbf{p})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{b}(\mathbf{p})] + p_0[\hat{b}(\mathbf{k})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{b}(\mathbf{k})]\rangle = 0, \tag{149}$$

we infer that

$$\frac{1}{2}\langle\hat{a}(\mathbf{p})\hat{b}(\mathbf{k}) + \hat{b}(\mathbf{k})\hat{a}(\mathbf{p})\rangle = g(\mathbf{p},\mathbf{k})\delta(p_0-k_0),$$
$$g(\mathbf{p},\mathbf{k}) = -g(\mathbf{k},\mathbf{p}). \tag{150}$$

By the same way if we assume that the state $\hat{\rho}$ is also invariant with respect to the spatial translations then

$$\langle\hat{a}(\mathbf{p})\hat{a}(\mathbf{k})\rangle = \langle\hat{b}(\mathbf{p})\hat{b}(\mathbf{k})\rangle = f(p)\delta(\mathbf{p}-\mathbf{k}),$$
$$\frac{1}{2}\langle\hat{a}(\mathbf{p})\hat{b}(\mathbf{k}) + \hat{b}(\mathbf{k})\hat{a}(\mathbf{p})\rangle = 0. \tag{151}$$

Thus we have for the dotted terms in formula (143)

$$\int\frac{d\mathbf{p}}{(2\pi)^{2d-2}}\frac{f(p)}{p_0}\cos[p_\mu(x-y)^\mu], \tag{152}$$

where one should bear in mind that the propagator expressed in terms of the operators $\hat{a}$ and $\hat{b}$ looks like

$$G(x,y) = \int\frac{d\mathbf{p}d\mathbf{k}}{(2\pi)^{2d-2}\sqrt{p_0k_0}}\langle[\hat{a}(\mathbf{p})\cos(p_\mu x^\mu) + \hat{b}(\mathbf{p})\sin(p_\mu x^\mu)] \times[\hat{a}(\mathbf{k})\cos(k_\mu y^\mu) + \hat{b}(\mathbf{k})\sin(k_\mu y^\mu)]\rangle, \tag{153}$$

provided $x^0 > y^0$. The additional requirement of an invariance of the state with respect to the Lorentz group yields that the function $f(p)$ contributes to the integral (152) as a constant. Consequently, this integral is proportional to the so-called Hadamard function [68],

$$G^{(1)}(x-y) := \int\frac{d\mathbf{p}\cos[p_\mu(x-y)^\mu]}{(2\pi)^{d-1}p_0}. \tag{154}$$

The Hadamard function does not vanish for the points x and y separated by a spatial interval. Hence if we want to obtain a causal theory we have to set the proportionality constant to zero that leaves us with the propagator proportional to the symmetric Green function.

The above considerations prove that the propagator proportional to the symmetric Green function is unique for the stochastically deformed model (131) under the causality condition and the requirement of the Poincaré invariance of the state. Explicitly such a pure state $\hat{\rho}=|\Psi\rangle\langle\Phi|$ can be specified as

$$|\Psi\rangle\langle\Phi| = \prod_{\mathbf{p}}|\Psi_{\mathbf{p}}\rangle\langle\Phi_{\mathbf{p}}|, \tag{155}$$

where

$$[\hat{a}^2(\mathbf{p}) + \hat{b}^2(\mathbf{p})]|\Psi_{\mathbf{p}}\rangle = \langle\Phi_{\mathbf{p}}|[\hat{a}^2(\mathbf{p}) + \hat{b}^2(\mathbf{p})] = 0. \tag{156}$$

In particular this implies

$$\hat{\mathcal{P}}_\mu|\Psi\rangle = \langle\Phi|\hat{\mathcal{P}}_\mu = 0. \tag{157}$$

The propagator (141) naturally appears when one computes, for example, the transition probability from the state $|\Psi_1\rangle\langle\Phi_1|$ to the state $|\Psi_2\rangle\langle\Phi_2|$,

$$\langle\Phi_2|\hat{U}|\Psi_1\rangle\langle\Phi_1|\hat{U}^{-1}|\Psi_2\rangle, \tag{158}$$

where $\hat{U}$ is the evolution operator over an infinite time and the states $|\Psi_1\rangle\langle\Phi_1|$ and $|\Psi_2\rangle\langle\Phi_2|$ are obtained from the ground state (155) by the action of the operators $\hat{a}$ and $\hat{b}$. The perturbation techniques is in a perfect analogy with such a procedure in quantum field theory. In our case the so-called closed-time-path formalism (see, e.g., [60,69] and references therein) seems to be especially useful for calculations. If the classical current is introduced into the model (131) then the solution (136) of the Heisenberg equations is the general solution of the homogeneous equation only. A particular solution of the inhomogeneous equation with current is given by a convolution of the retarded Green function with this classical current. In that case it is the convolution of the retarded Green function with current that is the average field over the ground state (155).

2. Electromagnetic field

As usual we start from the classical Maxwell action [70]

$$S[A] = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x[\dot{A}_i^2 - 2\partial_i A_0 \dot{A}_i + (\partial_i A_0)^2 - (\partial_i A_j)^2 + \partial_i A_j \partial_j A_i], \tag{159}$$

where $F=dA$ is the strength tensor of the electromagnetic field. Making the Legendre transformations on $\dot{A}_i$ only and introducing the canonical momenta

$$\pi_i = \frac{\partial L}{\partial \dot{A}^i} = \partial_i A_0 - \dot{A}_i, \tag{160}$$

we arrive at the Hamiltonian action

$$S_H = \int d^4x\left(\pi_i\dot{A}^i - \frac{1}{2}[\pi_i^2 + (\partial_i A_j)^2 - \partial_i A_j\partial_j A_i] + A_0\partial_i\pi^i\right). \tag{161}$$

As we see the time component of the gauge fields A_μ is merely the Lagrange multiplier to the Gauss law constraint. The expression in square brackets with 1/2 is the density of the physical Hamilton function H_0. Thus we have a model with first class constraints.

According to the standard Batalin-Fradkin-Vilkovisky quantization scheme for the first class constrained models we introduce the canonically conjugate ghost pairs (c,P) and $(\bar{c},\bar{P})$,

$$[\hat{c}(\mathbf{x}),\hat{P}(\mathbf{y})] = [\hat{\bar{c}}(\mathbf{x}),\hat{\bar{P}}(\mathbf{y})] = \hbar\,\delta(\mathbf{x}-\mathbf{y}),$$

$$\text{gh}\ c = \text{gh}\ \bar{c} = -\text{gh}\ P = -\text{gh}\ \bar{P} = 1, \tag{162}$$

where square brackets denote graded commutators. Besides we introduce the canonical momentum π_0, gh $\pi_0=0$, to the Lagrange multiplier A_0. Then we construct the BRST charge

$$\hat{\Omega} = \int d\mathbf{x}(\hat{c}\partial_i\hat{\pi}_i + \hat{\bar{c}}\hat{\pi}_0), \quad [\hat{\Omega},\hat{\Omega}] = 2\hat{\Omega}^2 = 0, \quad \text{gh}\ \hat{\Omega} = 1. \tag{163}$$

Physical observables and the Hamiltonian should commute with the BRST charge. In our case the Hamiltonian $\hat{H}_0$ obviously satisfies this requirement. The gauge fixing fermion is taken in the form

$$\hat{\psi} = \int d\mathbf{x}\left[\hat{P}\hat{A}_0 + \hat{\bar{P}}\left(\partial_i\hat{A}_i - \frac{\alpha}{2}\hat{\pi}_0\right)\right], \quad \text{gh}\ \hat{\psi} = -1, \tag{164}$$

where α is an arbitrary constant.

Therefore the gauge fixed Hamiltonian looks like

$$\hat{H} := \hat{H}_0 + [\hat{\Omega},\hat{\psi}] = \int d\mathbf{x}\left\{\frac{1}{2}[\hat{\pi}_i^2 + (\partial_i\hat{A}_j)^2 - \partial_i\hat{A}_j\partial_j\hat{A}_i] - \hat{A}_0\partial_i\hat{\pi}^i - \hat{\pi}_0\left(\partial_i\hat{A}^i + \frac{\alpha}{2}\hat{\pi}_0\right) - \hat{\bar{c}}\hat{P} + \partial_i\hat{c}\partial_i\hat{\bar{P}}\right\}, \tag{165}$$

and commutes with the BRST charge as well. The corresponding Heisenberg equations read as

$$\dot{\hat{A}}_i = \partial_i\hat{A}_0 - \hat{\pi}_i, \quad \dot{\hat{A}}_0 = \partial_i\hat{A}_i - \alpha\hat{\pi}_0, \quad \dot{\hat{c}} = \hat{\bar{c}}, \quad \dot{\hat{P}} = -\Delta\hat{\bar{P}},$$

$$\dot{\hat{\pi}}_i = \partial_i\partial_j\hat{A}_j - \Delta\hat{A}_i - \partial_i\hat{\pi}_0, \quad \dot{\hat{\pi}}_0 = -\partial_i\hat{\pi}_i, \quad \dot{\hat{\bar{c}}} = \Delta\hat{c}, \quad \dot{\hat{\bar{P}}} = -\hat{P}. \tag{166}$$

As expected the ghosts dynamics are split of the fields dynamics and we do not enlarge on them henceforth. Combining the Heisenberg equations it is not difficult to obtain

$$\Box\hat{A}_0 = (\alpha-1)\partial_i\hat{\pi}_i, \quad \Box\hat{A}_i = (1-\alpha)\partial_i\hat{\pi}_0, \quad \Box\hat{\pi}_i = 0. \tag{167}$$

In the Feynman gauge $\alpha=1$ the Heisenberg equations on the fields reduce to the wave equations and we can use all the results of the preceding example regarding the model of a scalar field. The physical states are those containing solely transverse photons.

Hence in the Feynman gauge the Poincaré-invariant causal propagator of the fields A_μ becomes

$$G(x-y) = \langle T\{\hat{A}_\mu(x)\hat{A}_\nu(y)\}\rangle = \hbar\,\eta_{\mu\nu}\bar{G}(x-y),$$

$$\Box\bar{G}(x) = \delta^d(x), \tag{168}$$

where $\bar{G}(x)$ is the symmetric Green function, what coincides with the well-known result of [29] in the case of a transparent medium $\varepsilon=\mu=1$. The correlators of the electromagnetic fields $F_{\mu\nu}$ are straightforwardly obtained from Eq. (168).

IV. CONCLUDING REMARKS

There are, of course, many important open problems regarding the procedure advocated here. We mention a few of them only. Apart from the study both analytical and numerical of solutions of the derived relativistic equations we distinguish a more detailed investigation of the model with nonlinear phase space. In spite of the fact that we considered several models of this kind their *a priori* interpretation remains unclear. For instance, in studying the stochastically deformed model of a relativistic particle with Hamilton function quadratic in momenta and with a nonlinear phase space, we found after constructing a path-integral representation of the transition probability that the probability distribution function of the random force is non-Gaussian. It is desirable to obtain a simple method to discover the noise distribution from the initial classical model. Furthermore we only touched the problem of stochastic deformation of field theories. The next step is to consider interacting models and their renormalization properties. It is also interesting to investigate the structure of the gauge transformations and corresponding functional gauge fields associated with the changing of the phase S in the stochastic field theory framework. Notice that the gauged formulation of stochastic mechanics developed in this paper allows us to represent it in terms of sections of vector bundles analogously to quantum mechanics. For the Schrödinger equations of the form (37) the pair of functions $\varphi(x)$ and $\psi(x)$ describing a pure state of a stochastic system can be viewed as a section (written in the light cone coordinates) of a vector bundle over the space-time with the typical fiber $\mathbb{R}^2$ and the structure group $SO(1,1)$. Then the states should be described by timelike or isotropic sections to guarantee the non-negativity of probability density functions. The gauge fields are connections on this vector bundle. A similar construction on the Whitney sum of tangent and cotangent bundles $TM \oplus T^*M$ is called an almost generalized product structure (see, for an introduction, [71]) that is a real analog of an almost generalized complex structure [72].

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